9.10 Taylor's Theorem (Lagrange Error Bound)

If function f is differentiable through order n+1 in an interval containing the center x=c, then for each x=a in the interval, there exists a number x=z between a & c such that:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(a - c) + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where the remainder $R_n(x)$ is given by $R_n(x) = \left|\frac{|f^{(n+1)}(z)|}{(n+1)!}(x - c)^{n+1}\right|$, called the **Lagrange Remainder, or Lagrange Error Bound.** When applying Taylor's Theorem, we would not expect to be able to find the exact value of z. Rather, we are merely interested in a "safe" upper bound (max value) for $|f^{(n+1)}(z)|$ from which we will be able to tell how large the remainder $R_n(x)$ is.

You need 4 things: f(x) = ? c = ? x = ? n+1 = ?

1. (calculator) Let f be a function with 5 derivatives on the interval [2, 3]. Assume that $f^{(5)}(z) < .2$ for all x in [2, 3] & that a fourth-degree Taylor polynomial, $P_4(x)$, for f at c = 2 is used to estimate $f(3)$.

   a. How accurate is this approximation? Give 4 decimal places.
   b. Suppose that $P_4(3) = 1.763$. Use your answer from (a) to find an interval in which $f(3)$ must reside.
   c. Could $f(3)$ equal 1.761? Explain.

2. (calculator) The function f has derivative of all orders for all real numbers x. Assume that *f(2) = 6, f ' (2) = 4, f " (2) = -7, f '"(2) = 8.*

   a. Write the 3rd degree Taylor polynomial for f about x = 2 and use it to approximate $f(2.3)$. Give three decimal places.
   b. The 4th derivative of f satisfies the inequality $|f^4(x)| \le 9$ for all x in the closed interval [2, 2.3]. Use the Lagrange error bound on the approximation of *f(2.3)* found in part (a) to find an interval [a, b] such that a$\le f(2.3) \le b$. Give three decimal places.
   c. Based on the information above, could *f(2.3)* equal 6.922? Explain.

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   d. How accurate is this approximation? Give 4 decimal places.
   e. Suppose that $P_4(3) = 1.763$. Use your answer from (a) to find an interval in which $f(3)$ must reside.
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